arXiv:0909.4835v1 [math.QA] 26 Sep 2009

A Vertex Algebra Commutant for the $\beta\gamma$-System and Howe pairs

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September 26, 2009

Abstract: Analogue to commutants in the theory of associative algebras, one can construct a new subalgebra of vertex algebra known as a vertex algebra commutant. In this paper, for the adjoint representation V of Lie algebra $sl(2,\mathbb{C})$, we describe a commutant of $\beta\gamma$- System $S(V)$ by giving its generators, moreover, we get a new Howe pair of vertex algebras.
Keywords: Vertex Algebra Commutant, $\beta\gamma$-System, ∂-ring, Hilbert series
MSC(2000): 17B69 & 81T40

1 Introduction

Let W be a vertex algebra, and U be its subalgebra, one can construct a new subalgebra, which is known as the commutant $Com(U,W)$ of U in W. In fact, this construction is a generalization of the coset construction in conformal field theory due to Kac-Peterson [2] and Goddard-Ken-Olive [3], and was introduced by I. Frenkel and Zhu in [4] in mathematics. It is an analogue to the ordinary commutant construction in associative algebra theory.

In order to describe vertex algebra commutant $Com(U,W)$ more clearly, we expect to find its generator set and the corresponding relations. But we don't know whether the commutant $Com(U,W)$ is generated finitely, and how to find its generators. These are non-trivial problems. It's obviously that $U \subset Com(Com(U,W),W)$. If $U = Com(Com(U,W),W)$, the pair U and $(Com(U,W))$ is called a Howe pair([5]). As in associate algebra, it should have applications in vertex algebra theory. Since up to now, we know little about these new objects, it should be useful to construct new more examples in the vertex algebra category.

According to the paper [6], the notion of a commutative circle algebra is abstractly equivalent to the notion of a vertex algebra, and there are the re-

*Supported in part by NSFC with grant Number 10471034 & 10971071 and Provincial Foundation of Innovative Scholars of Henan.

lated correspondence between two theories. We shall refer to a commutative circle algebra simply as a vertex algebra throughout this paper.

Let $\mathbf{g}$ be a finite dimensional complex semisimple Lie algebra, V be a finite dimensional complex $\mathbf{g}-$module via the Lie algebra homomorphism $\rho : \mathbf{g} \longrightarrow EndV$. The representation ρ induces a vertex algebra homomorphism $\widehat{\rho}$ from $\emptyset(\mathbf{g}, B)$ to $S(V)$, where B is the bilinear form $B(u,v) = -Tr(\rho(u)\rho(v))$ and $S(V)$ is the $\beta\gamma-$system which was introduced in [1]). If V admits a symmetric invariant bilinear form B', there is a vertex algebra homomorphism $\widehat{\psi} : \emptyset(sl(2,\mathbb{C}), -\frac{dimV}{8}K) \longrightarrow S(V)^{\Theta_+}$, denote by $\mathcal{A} = \widehat{\psi}(\emptyset(sl(2,\mathbb{C}), -\frac{n}{8}K))$(cf. §2 in [8] in details). In [14], Zhu introduced a functor which assigns each vertex algebra W to an associative algebra $A(W)$(known as the Zhu algebra of W). From [4], we know that the associative algebra $A(\emptyset(\mathbf{g}, B))$ is isomorphic to the universal enveloping algebra $U(\mathbf{g})$. Let $D(V)$ be the Weyl algebra of polynomial differential operators of V. It is well known that $A(S(V))$ is isomorphic to $D(V)$. Just like the classical commutant theory of associative algebras, how to describe the commutant $S(V)^{\Theta_+}$ of $\Theta = \widehat{\rho}(\emptyset(\mathbf{g}, B))$ in $S(V)$ is an important problem.

As we know, there are some results on the descriptions of the commutant $S(V)^{\Theta_+}$. In [8, 11], B. Lian and Linshaw studied the vertex algebra and invariant theory, and reduced the problem of describing $S(V)^{\Theta_+}$ to a problem in commutative algebra. They single out a certain category $\mathfrak{R}$ of vertex algebras equipped with a $\mathbb{Z}_{\geq 0}-$ filtration such that the associated graded objects are $\mathbb{Z}_{\geq 0}-$graded $\partial-$ rings. All vertex algebras of the form $S(V)$ and $\emptyset(\mathbf{g}, B)$ belong to the category $\mathfrak{R}$, and so are their subalgebras $\widehat{\rho}(\emptyset(\mathbf{g}, B))$ and $S(V)^{\Theta_+}$. In particular, the assignment $W \longmapsto gr(W)$ is a functor from $\mathfrak{R}$ to the category $\mathcal{R}$ of $\mathbb{Z}_{\geq 0}-$ graded $\partial-$rings, and the main object of study $S(V)^{\Theta_+}$ is sent to the $\partial-$ ring $gr(S(V)^{\Theta_+})$. It's lucky that the reconstruction property of the category $\mathcal{R}$ tells us that if one can find a generator set of $gr(S(V)^{\Theta_+})$, then he can construct a generator set of $S(V)^{\Theta_+}$. However, describing generators of $gr(S(V))^{\Theta_+}$ is much easier than that of $gr(S(V)^{\Theta_+})$ in the invariant theory. Moreover, there is a canonical injection $\Gamma : gr(S(V)^{\Theta_+}) \longrightarrow gr(S(V))^{\Theta_+}$, if Γ is surjective, the generator set of $gr(S(V))^{\Theta_+}$ can be regarded as the generator set of $gr(S(V)^{\Theta_+})$, hence, one need to find the generator set of $gr(S(V))^{\Theta_+}$. As an example in [11], taking $\mathbf{g} = V = sl(2,\mathbb{C})$, Linshaw studied the subalgebras $gr(S_\beta(V)^{\Theta_+})$ and $gr(S_\gamma(V)^{\Theta_+})$ of $gr(S(V)^{\Theta_+})$, and gave a complete description of vertex algebras $S_\beta(V)^{\Theta_+}$ and $S_\gamma(V)^{\Theta_+}$. Moreover, he showed that $\mathcal{A}$ is isomorphic to the current algebra $\emptyset(sl(2,\mathbb{C}), -\frac{3}{8}K)$. In terms of $V = \mathbf{g} = sl(2,\mathbb{C})$, in [8], the authors used the properties of Gröbner bases to prove that $S(V)^{\mathcal{A}_+} = \Theta$ and obtained a Howe pair $(\Theta, S(V)^{\Theta_+})$ in $S(V)$, based on above approach to describe $S(V)^{\Theta_+}$. About the case that $\mathbf{g}$ is abelian Lie algebra acting diagonally on a vector space V, Linshaw found a finite set of generators for $S(V)^{\Theta_+}$, and showed that $S(V)^{\Theta_+}$ is a simple vertex algebra and a member

of Howe pair ([10]). More generally, if $\mathbf{g} = sl(n,\mathbb{C}), so(n,\mathbb{C}), sp(2n,\mathbb{C})$ and V are the copies of standard representations of $\mathbf{g}$, the authors used tools from commutative algebra and algebraic geometry, in particular, the theory of jet schemes, to describe $S(V)^{\Theta_+}$ and give some Howe pairs in vertex algebra context([12]).

In this paper, based on the theory of vertex algebras and $\partial-$ rings in [6, 8, 9, 11, 12], we also study the case of $V = \mathbf{g} = sl(2,\mathbb{C})$. Under the related properties of Hilbert series, we find all finite generators of invariant ring $gr(S(V))^{\Theta_+}$, and then we give the finite generator set of $S(V)^{\Theta_+}$ explicitly. Moreover, we get a new Howe pair $(\mathcal{A}, S(V)^{\mathcal{A}_+})$ in $S(V)$. Here is detailed outline of the paper. Firstly, the action of $\mathbf{g}\otimes\mathbb{C}[t]$ on $gr(S(V))$ induced by the adjoint representation V, forms the invariant subalgebra $gr(S(V))^{\mathbf{g}\otimes\mathbb{C}[t]}$. Denote by $P = gr(S(V))$, by the theorem 5.9 in [12], we get the $\partial-$ ring $P^{\mathbf{g}\otimes\mathbb{C}[t]}$ is generated by $P_0^{\mathbf{g}}$. In particular, the finite generator set of ring $P_0^{\mathbf{g}}$ is also the generator set of $P^{\mathbf{g}\otimes\mathbb{C}[t]}$ as a $\partial-$ ring. Secondly, we find finite generators and the corresponding relations of invariant ring $P_0^{\mathbf{g}}$ by solving the Hilbert series of $V\oplus V^*$. Since the embeddings $gr(\mathcal{A}) \hookrightarrow gr(S(V)^{\Theta_+}) \hookrightarrow gr(S(V))^{\mathbf{g}\otimes\mathbb{C}[t]}$ are both surjections, by the reconstruction property of $\partial-$ rings, we give the finite generator set of $S(V)^{\Theta_+}$. Moreover, we also get a new Howe pair $(\mathcal{A}, S(V)^{\mathcal{A}_+})$.

2 Vertex Algebras and Some Examples

In this section, we give a summary of vertex algebras for this paper can be read easily. Please refree to papers [6, 8, 11, 12] for the details. We shall use such vertex algebra notions throughout this paper.

Let V be a vector space over $\mathbb{C}$, and z, w be the formal variables. Denote the space of all linear maps $V \to V((z))$ by $EndV((z))$, where

$$V((z)) := \left\{ \sum_{n\in\mathbb{Z}} v(n)z^{-n-1} \mid v(n) \in V, v(n) = 0 \text{ for } n \gg 0 \right\}.$$

Then each $v \in EndV((z))$ can be uniquely expressed as a formal series $v = v(z) := \sum\limits_{n\in\mathbb{Z}} v(n)z^{-n-1}$, where $v(n) \in EndV$.

On the space $EndV((z))$, for $n \in \mathbb{Z}$, one can define n-th circle products as follows: For $u, v \in EndV((z))$, n-th circle products is defined by

$$u(w)\circ_n v(w) = Res_{z=0}(u(z)v(w)\ell_{|z|>|w|}(z-w)^n - v(w)u(v)\ell_{|w|>|z|}(z-w)^n).$$

Here $\ell_{|z|>|w|}f(z,w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function $f(z,w)$ in the region $|z| > |w|$.

The non-negative circle products are connected through the operator product expansion (OPE) formula. For $u, v \in EndV((z))$, there are

$$u(z)v(w) = \sum_{n\geq 0} u(w)\circ_n v(w)(z-w)^{-n-1} + :u(z)v(w):,$$

or it is written as $u(z)v(w) \sim \sum\limits_{n\geq 0} u(w)\circ_n v(w)(z-w)^{-n-1}$, where $\sim$ means equal modulo the term $: u(z)v(w) :$. Here, $: u(w)v(w) :$ is a well defined element of $EndV((z))$, called the Wick product of u and v, and there is $: u(w)v(w) := u \circ_{-1} v$. The other circle products are related to this by $n!u(z)_{-n-1}v(z) =: \partial^n u(z)v(z) :$ for non-negative integers n, where ∂ denotes the formal differentiation operator $\frac{d}{dz}$. For $u \in EndV((z))$, there is $1 \circ_n u = \delta_{n,-1}u$ for all $n \in \mathbb{Z}$; $u \circ_n 1 = \delta_{n,-1}u$ for $n \geq -1$.

A linear subspace $U \subset EndV((z))$ containing 1 which is closed under the circle products will be called a circle algebra. In particular, U is closed under ∂ since $\partial u = u \circ_{-2} 1$. Let U be a circle algebra, a subset $S = \{u_i | i \in I\}$ of U is called to generate U if any element of U can be written as a linear combination of non-associative words in the letters $u_i, \circ_n$ for $i \in I$ and $n \in \mathbb{Z}$. It is said that S strongly generates U if any element of U can be written as a linear combination of words in the letters $u_i, \circ_n$ for $n < 0$. Equivalently, U is spanned by the collection $\{: \partial^{k_1} u_{i_1}(z)\partial^{k_2} u_{i_2}(z)\cdots\partial^{k_m} u_{i_m}(z) : | k_1, k_2, \cdots, k_m \geq 0\}$.

Definition 2.1. We say that $u, v \in EndV((z))$ circle commute if $(z-w)^N[u(z), v(w)] = 0$ for some $N \geq 0$. Here $[\ ,\ ]$ denotes the bracket. If N can be choose to be zero. we say that u, v commute. A circle algebra is said to be commutative if its elements pairwise circle commute.

The notion of a commutative circle algebra is abstractly equivalent to the notion of a vertex algebra. For any commutative circle algebra U, define

$$\begin{aligned} &\pi : U \longrightarrow (EndU)((t)) \\ &u \longmapsto \pi(u) : \pi(u)v = \sum_{n\in\mathbb{Z}} (u \circ_n v)t^{-n-1}, \text{ for } \forall v \in U. \end{aligned}$$

Then π is an injective circle algebra homomorphism, and the quadruple of structure $(U, \pi, 1, \partial)$ is a vertex algebra. Conversely, if $(V, Y, 1, D)$ is a vertex algebra, the collection $\{Y(v, z)| v \in V\} \subset EndV((z))$ is a commutative circle algebra.

Next, to study the theory of commutants of vertex algebras, we introduce to two examples of vertex algebras.

Example 1(current algebra). Let $\mathbf{g}$ be a Lie algebra equipped with a symmetric invariant bilinear form B, and $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial algebra over $\mathbb{C}$ with one indeterminate t, affine Lie algebra $\widehat{\mathbf{g}} = \mathbf{g}\otimes\mathbb{C}[t, t^{-1}]\oplus \mathbb{C}c$ with bracket

$$[u \otimes t^n, v \otimes t^m] = [u, v] \otimes t^{m+n} + nB(u, v)\delta_{m+n,0}c,$$

where c is the center of $\widehat{\mathbf{g}}$.

Set $deg(u\otimes t^n) = n, deg(K) = 0$, then $\widehat{\mathbf{g}}$ is equipped with the $\mathbb{Z}-$ grading structure. Let $\widehat{\mathbf{g}}_{\geq 0} \subset \widehat{\mathbf{g}}$ be the subalgebra of elements of non-negative degree, and let $N(\mathbf{g}, B) = \mathfrak{U}(\widehat{\mathbf{g}}) \otimes_{\widehat{\mathbf{g}}_{\geq 0}} \mathbb{C}$ be the induced $\widehat{\mathbf{g}}-$module, where $\mathbb{C}$ is the

1-dimensional $\widehat{\mathbf{g}}_{\geq 0}$−module on which $\mathbf{g} \otimes \mathbb{C}[t]$ acts by zero and c acts by 1. For each $u \in \mathbf{g}$, let $u(n)$ be the linear operator on $N(\mathbf{g}, B)$ representing $u \otimes t^n$, and put $u(z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1} \in End(N(\mathbf{g}, B))((z))$. The collection $\{u(z) | u \in \mathbf{g}\}$ generates a vertex algebra in $End(N(\mathbf{g}, B))((z))$, which we denote by $\emptyset(\mathbf{g}, B)$([4, 6, 8, 11]). For any $u, v \in \mathbf{g}$, the vertex operators $u(z), v(z) \in End(N(\mathbf{g}, B))((z))$ satisfy OPE

$$u(z)v(w) \sim B(u, v)(z - w)^{-2} + [u, v](w)(z - w)^{-1}. \tag{2.1}$$

Denote $\mathbf{1}$ by the vacuum vector $1 \otimes 1 \in N(\mathbf{g}, B)$, then there is the conclusion

Lemma 2.2. *The creation map $\chi : \emptyset(\mathbf{g}, B) \longrightarrow N(\mathbf{g}, B)$ sending $u(z) \longmapsto u(-1)\mathbf{1}$ is an isomorphism of $\emptyset(\mathbf{g}, B)$−modules.*

If $\mathbf{g}$ is simple, for $\lambda \in \mathbb{C}, \lambda \neq -\frac{1}{2}$, $\emptyset(\mathbf{g}, B)$ has a conformal element $L_{\emptyset}(z) = \frac{1}{2\lambda+1} \sum_{i=1}^{dim\mathbf{g}} : u_i(z)u_i(z) :$, where $\{u_i | i = 1, 2, \cdots, dim\mathbf{g}\}$ is an orthonormal basis of $\mathbf{g}$ with respect to the killing form K.

Example 2($\beta\gamma$−system). Let V be a finite dimensional vector space. Regarding $V \oplus V^*$as an abelian Lie algebra, affine Lie algebra $\eta(V) = (V \oplus V^*) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\tau$ with bracket

$$[(x, x') \otimes t^n, (y, y') \otimes t^m] = (\langle y', x \rangle - \langle x', y \rangle)\delta_{m+n,0}\tau, \tag{2.2}$$

for $x, y \in V$, and $x', y' \in V^*$. Let $\sigma \subset \eta(V)$ be the subalgebra generated by $\tau, (x, 0) \otimes t^n$, and $(0, x') \otimes t^m$ for $n \geq 0, m > 0$. Let $\mathbb{C}$ be the 1-dimensional σ−module on which $(x, 0) \otimes t^n$ and $(0, x') \otimes t^m$ act trivially and the central element τ acts by the identity. Denote the linear operators representing $(x, 0) \otimes t^n, (0, x') \otimes t^n$ on the induced module $\mathfrak{U}(\eta(V)) \otimes_{\mathfrak{U}(\sigma)} \mathbb{C}$ by $\beta^x(n), \gamma^{x'}(n - 1)$, respectively, for $n \in \mathbb{Z}$. The power series

$$\beta^x(z) = \sum_{n \in \mathbb{Z}} \beta^x(n) z^{-n-1}, \gamma^{x'}(z) = \sum_{n \in \mathbb{Z}} \gamma^x(n) z^{-n-1} \tag{2.3}$$

generate a vertex algebra $S(V)$ in $End(\mathfrak{U}(\eta(V)) \otimes_{\mathfrak{U}(\sigma)} \mathbb{C})((z))$, called $\beta\gamma$−system([1]), and the generators satisfy OPE

$$\beta^x(z)\gamma^{x'}(w) \sim < x', x > (z - w)^{-1}, \beta^x(z)\beta^y(w) \sim 0, \gamma^{x'}(z)\gamma^{y'}(w) \sim 0. \tag{2.4}$$

Suppose that V is a n-dimensional $\mathbf{g}$−module via $\rho : \mathbf{g} \longrightarrow EndV$, where $\mathbf{g}$ is a finite dimensional Lie algebra. There is the following relation between these two vertex algebras.

Lemma 2.3. *([8]) The above map ρ induces a vertex algebra homomorphism $\widehat{\rho} : \emptyset(\mathbf{g}, B) \longrightarrow S(V)$, where B is the symmetric invariant bilinear form $B(u, v) = -Tr(\rho(u)\rho(v))$ for $u, v \in \mathbf{g}$.*

Here, let$\{x_1, x_2, \cdots, x_n\}$ be a basis of V and $\{x'_1, x'_2, \cdots, x'_n\}$ be dual basis of V^*, the vertex algebra homomorphism $\widehat{\rho}$ send $u(z)$ to

$$\widehat{u}(z) = -\sum_{i=1}^{n} : \beta^{\rho(u)(x_i)}(z)\gamma^{x'_i}(z) :, \forall u \in \mathbf{g}. \tag{2.5}$$

And there is the following operator product expansions

$$\widehat{u}(z)\widehat{v}(w) \sim B(u,v)(z-w)^{-2} + \widehat{[u,v]}(w)(z-w)^{-1}, \tag{2.6}$$

for $u, v \in \mathbf{g}$. Incidently, $S(V)$ has a conformal element $L_S(z) = \sum_{i=1}^{dimV} : \beta^{x_i}(z)\partial\gamma^{x'_i}(z) :$.

Analogous to the commutant construction in the theory of associative algebra, one can has a way to construct vertex subalgebras of a vertex algebra, known as commutant subalgebras.

Definition 2.4. Let W be a vertex algebra and U be any subset of W. The commutant of U in W, denote by $Com(U, W)$, is defined to be the set of vertex operators $v(w) \in W$ commuting strictly with each element of U, that is, $[u(z), v(w)] = 0$ for all $u(z) \in U$.

In terms of above circle product, $[u(z), v(w)] = 0$ if and only if $u(z) \circ_n v(w) = 0$ for all $n \geq 0$, so there is also

$$Com(U, W) = \{v(z) \in W | u(z) \circ_n v(z) = 0, \forall u(z) \in U, n \geq 0\} \tag{2.7}$$

Regard W^{U_+} as the subalgebra of invariants in W under the action of U. If Θ is a vertex algebra homomorphic image of a current algebra $\emptyset(\mathbf{g}, B)$, W^{Θ_+} is just the invariant space $W^{\mathbf{g}\otimes\mathbb{C}[t]}$. According to the formula (2.7), the action of Θ on W is induced by the non-negative circle products, hence we can write $Com(\Theta, W)$ as W^{Θ_+}.

According to the vertex algebra homomorphism $\widehat{\rho}$ in Lemma 2.3, there is the following definition

Definition 2.5. Let Θ be the subalgebra $\widehat{\rho}(\emptyset(\mathbf{g}, B)) \subset S(V)$. The commutant algebra $S(V)^{\Theta_+} = Com(\widehat{\rho}(\emptyset(\mathbf{g}, B)), S(V))$ is called the algebra of invariant chiral differential operators on V.

For a simple Lie algebra $\mathbf{g}$ and $B(u,v) = \lambda K(u,v)$ for $\lambda \in \mathbb{C}, \lambda \neq -\frac{1}{2}$, $S(V)^{\Theta_+}$ has the conformal elements $L(z) = L_S(z) - \widehat{\rho}(L_{\emptyset}(z))$.

Lemma 2.6. *([8]) The conformal weight-zero subspace* $S(V)_0^{\Theta_+} \subset S(V)^{\Theta_+}$ *coincides with the classical invariant ring* $Sym(V^*)^{\mathbf{g}}$.

Let V be a n-dimensional vector space, $D(V)$ be the Weyl algebra of polynomial differential operators of V, then $D(V)$ has generators $\beta^x, \gamma^{x'}$, which are linear in $x \in V, x' \in V^*$, and satisfies the commutation relations $[\beta^x, \gamma^{x'}] =< x', x >$. If V is a n-dimensional $\mathbf{g}-$module via $\rho : \mathbf{g} \longrightarrow EndV$, there is an induced action ρ^* of $\mathbf{g}$ on $D(V)$, moreover, $\mathbf{g}$ acts on $D(V)$ by derivations of degree 0, and we have $gr(D(V)^{\mathbf{g}}) = gr(D(V))^{\mathbf{g}} = Sym(V \oplus V^*)^{\mathbf{g}}$. The action of $\mathbf{g}$ on $D(V)$ can be realized by inner derivations. We have a Lie algebra homomorphism $\tau : \mathbf{g} \longrightarrow D(V)$ given in chosen basis by

$$\tau = -\sum_{i=1}^{n} \beta^{\rho(u)(x_i)} \gamma^{x'_i}, \tag{2.8}$$

which can be extended to a map $\mathfrak{U}(\mathbf{g}) \longrightarrow D(V)$, and the action of $u \in \mathbf{g}$ on $v \in D(V)$ is given by $\rho^*(v) = [\tau(u), v]$. Thus $D(V)^{\mathbf{g}} = Com(\tau(\mathfrak{U}(\mathbf{g}), D(V))$.

Let V be a n-dimensional $\mathbf{g}$- module equipped with a symmetric invariant bilinear form B'. If $\{x_1, x_2, \cdots, x_n\}$ is an orthonormal basis of V with respect to B', and $\{x'_1, x'_2, \cdots, x'_n\}$ is the corresponding dual basis of V^*, there are the following two lemmmas in [8, 11]

Lemma 2.7. *There is a Lie algebra homomorphism $\psi : sl(2, \mathbb{C}) \longrightarrow D(V)^{\mathbf{g}}$ given in an orthonormal basis with respect to B' by the formulas*

$$h \longmapsto \sum_{i=1}^{n} \beta^{x_i} \gamma^{x'_i}, e \longmapsto \frac{1}{2} \sum_{i=1}^{n} \gamma^{x'_i} \gamma^{x'_i}, f \longmapsto -\frac{1}{2} \sum_{i=1}^{n} \beta^{x_i} \beta^{x_i}, \tag{2.9}$$

where $\{e, f, h\}$ denote the standard generators of $sl(2, \mathbb{C})$, satisfying

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f.$$

Lemma 2.8. *The homomorphism $\psi : sl(2, \mathbb{C}) \longrightarrow D(V)^{\mathbf{g}}$ induces a vertex algebra homomorphism $\widehat{\psi} : \text{Ø}(sl(2, \mathbb{C}), -\frac{n}{8}K) \longrightarrow S(V)^{\Theta_+}$ by*

$$h(z) \longmapsto v^h(z) = \sum_{i=1}^{n} : \beta^{x_i}(z) \gamma^{x'_i}(z) : \tag{2.10}$$

$$e(z) \longmapsto v^e(e) = \frac{1}{2} \sum_{i=1}^{n} : \gamma^{x'_i}(z) \gamma^{x'_i}(z) : \tag{2.11}$$

$$f(z) \longmapsto v^f(z) = -\frac{1}{2} \sum_{i=1}^{n} : \beta^{x_i}(z) \beta^{x_i}(z) : \tag{2.12}$$

where K is the Killing form of $sl(2, \mathbb{C})$.

3 Category $\mathfrak{R}$ and $\partial-$ Rings

In this section, we introduce a certain category of vertex algebras and a category of $\partial-$ rings. Here we refer to the related theory of [8, 9, 11, 12, 13].

Let $\mathfrak{R}$ be the category of pairs (W, deg), where W is a vertex algebra equipped with a $\mathbb{Z}_{\geq 0}-$filtration

$$W_0 \subset W_1 \subset W_2 \subset \cdots, W = \bigcup_{k\geq 0} W_k,$$

such that $W_0 = \mathbb{C}$, and for $a \in W_k, b \in W_l$, there are

$$a \circ_n b \in W_{k+l}, \text{ for } n < 0, \tag{3.1}$$

$$a \circ_n b \in W_{k+l-1}, \text{ for } n \geq 0. \tag{3.2}$$

Here $W_k = 0$ for $k < 0$. A non-zero element $a(z) \in W$ is said to have degree d if d is the minimal integer for which $a(z) \in W_d$. Morphisms in $\mathfrak{R}$ are morphisms of vertex algebras which preserve the above filtration. Filtration on vertex algebras satisfying (3.1),(3.2) is induced by Haisheng Li in [9] known as good increasing filtration. If W possesses such a filtration, it follows that the associated graded object $gr(W) = \bigoplus_{k>0} W_k/W_{k-1}$ is a $\mathbb{Z}_{\geq 0}-$ graded associative, commutative algebra with a unit 1 under a product induced by the Wick product on W. Moreover, $gr(W)$ has a derivation ∂ of degree zero 0, and for each $a \in W_d$ and $n \geq 0$, operators $a\circ_n$ on W induces a derivation of degree $d-1$ on $gr(W)$. For each $d \geq 1$, we have the projection $\phi_d : W_d \longrightarrow W_d/W_{d-1} \in gr(W)$.

If $u, v \in gr(W)$ are homogeneous of degree r, s, respectively, and $u(z) \in W_r, v(z) \in W_s$ are vertex operators such that $\phi_r(u(z)) = u$ and $\phi_s(v(z)) = v$, it follows that $\phi_{r+s}(: u(z)v(z) :) = uv$.

Let $\mathcal{R}$ denote the category of $\mathbb{Z}_{\geq 0}-$graded commutative rings equipped with a derivation ∂ of degree zero, which is called $\partial-$rings.

The prominent feature of $\mathcal{R}$ is that vertex algebra $W \in \mathcal{R}$ has the following reconstruction property. We can write down a set of strong generators for W as a vertex algebra just by studying the ring structure of $gr(W)$. We say that the collection $\{a_i | i \in I\}$ generates $gr(W)$ as a $\partial-$ring if the collection $\{\partial^k a_i | i \in I, k \geq 0\}$ generates $gr(W)$ as a grading ring.

Lemma 3.1. *([8]) Let W be a vertex algebra in $\mathcal{R}$. Suppose that $gr(W)$ is generated as a $\partial-$ ring by a collection $\{a_i | i \in I\}$, where a_i is homogeneous of degree d_i, choose vertex operators $a_i(z) \in W_{d_i}$ such that $\phi_{d_i}(a_i(z)) = a_i$, then W is strongly generated by the collection $\{a_i(z) | i \in I\}$.*

Define $k = k(W, deg) = Sup\{j \geq 1 | W_r \circ_n W_s \subset W_{r+s-j}, \forall r, s, n \geq 0\}$, it follows that k is finite if and only if W is not abelian(cf. [8]). For two vertex algebras $\emptyset(\mathbf{g}, B)$ and $S(V)$, there are $k(\emptyset(\mathbf{g}, B), deg) = 1, k(S(V), deg) = 2$.

Lemma 3.2. *([8]) Let $(W, deg) \in \mathfrak{R}$, and $k = k(W, deg)$ be as above. For each $u(z) \in W$ of degree d and $n \geq 0$, the operator $u(z)\circ_n$ on W induces a homogeneous derivation $u(n)_{Der}$ on $gr(W)$ of degree $d-k$, defined on homogeneous elements v of degree r by*

$$u(n)_{Der}(v) = \phi_{r+d-k}(u(z) \circ_n v(z)). \tag{3.3}$$

Here $v(z) \in W$ is any vertex operator of degree r such that $\phi_r(v(z)) = v$.

Let $(W, deg) \in \mathfrak{R}$, $\mathcal{C}$ be a subalgebra of W which is a homomorphic image of a current algebra $\O(\mathbf{g}, B)$. Suppose that for each $u \in \mathbf{g}$, $u(z) \in \mathcal{C}$ has degree k, so that the derivations $\{u(n)_{Der}|n \geq 0\}$ on $gr(W)$ are homogeneous of degree 0. Then there is

Lemma 3.3. *([8]) The derivations $\{u(n)_{Der}|n \geq 0\}$ form a representation of $\mathbf{g} \otimes \mathbb{C}[t]$ on $gr(W)$. Moreover, the actions of $\mathbf{g} \otimes \mathbb{C}[t]$ on W and $gr(W)$ are compatible in the sense that for any $w(z) \in W$ of degree r, there are $u(n)_{Der}\phi_r(w(z)) = \phi_r \circ u(n)(w(z))$.*

Consider the invariant ring $gr(W)^{\mathcal{C}_+}$, since $gr(W)^{\mathcal{C}_+}$ is closed under ∂, then $W^{\mathcal{C}_+} \hookrightarrow W$ induces an injective ring homomorphism $gr(W^{\mathcal{C}_+}) \hookrightarrow gr(W)$ whose image clearly lies in $gr(W)^{\mathcal{C}_+}$. So we have a canonical injective homomorphism $\Gamma : gr(W^{\mathcal{C}_+}) \hookrightarrow gr(W)^{\mathcal{C}_+}$ as $\partial-$rings.

Let $W = S(V), \Theta = \widehat{\rho}(\O(\mathbf{g}, B))$, where $\mathbf{g}$ is semisimple and V is a finite dimensional $\mathbf{g}$-module. In the case, $deg(\widehat{u}(z)) = 2 = k$, so each $\widehat{u}(n)_{Der}$ is homogeneous of degree 0 and $gr(S(V)$ is a $\mathbf{g} \otimes \mathbb{C}[t]-$module by Lemma 3.3. Denote by $P = gr(S(V))$, and we write the images of $\partial^k \beta^x(z), \partial^k \gamma^{x'}(z)$ in P as β_k^x and $\gamma_k^{x'}$, respectively. The action of $\widehat{u}(n)_{Der}$ on the generators of P is given by

$$\widehat{u}(n)_{Der}(\beta_k^x) = C_k^n \beta_{k-n}^{\rho(u)(x)}; \widehat{u}(n)_{Der}(\gamma_k^{x'}) = C_{k-n}^n \gamma^{\rho^*(u)(x')}, \tag{3.4}$$

where $C_k^n = k(k-1)\cdots(k-n+1)$ for $n, k \geq 0$, $C_k^0 = 1, C_k^n = 0$ for $n > k$.

Next, we shall state a conclusion in [12], which plays an important role for this paper.

Let V be a linear representation of G (connected, reductive linear algebraic group over $\mathbb{C}$ with Lie algebra $\mathbf{g}$). Choose a basis $\{x_1, x_2, \cdots, x_n\}$ for V^*, so the regular function ring $\O(V) \cong \mathbb{C}[x_1, x_2, \cdots, x_n]$, and there is $P = gr(S(V)) = \mathbb{C}[\beta_k^x, \gamma_k^{x'}|x \in V, x' \in V^*, k \geq 0]$.

Lemma 3.4. *([12]) Let G be a connected, reductive algebraic group, and let V be a linear $G-$ representation such that $\O(V \oplus V^*)$ contains no invariant lines. Then $P^{\mathbf{g}\otimes\mathbb{C}[t]}$ is generated by*

$$P_0^G = \mathbb{C}[\beta_0^x, \gamma_0^{x'}|x \in V, x' \in V^*]^G \cong \O(V \oplus V^*)^G$$

as a $\partial-$ ring. In particular, if $\{f_1, f_2, \cdots, f_n\}$ generate P_0^G as a ring, then $\{f_1, f_2, \cdots, f_n\}$ generate $P^{\mathbf{g}\otimes\mathbb{C}[t]}$ as a $\partial-$ ring.

At the same times, we have the well-known theorem(cf.[13])

Lemma 3.5. *(Hilbert Theorem) If G is a connected, reductive algebraic group, then the invariants ring of polynomials $\mathbb{C}[x_1, x_2, \cdots, x_n]^G$ is finitely generated.*

For $\mathbf{g} = sl(2,\mathbb{C})$, we take $V = sl(2,\mathbb{C})$ the adjoint representation. Since special linear group $SL(2)$ is a connected, linear reductive algebraic group with the Lie algebra $sl(2,\mathbb{C})$. By Lemma 3.4, $P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is generated by $P_0^{sl(2,\mathbb{C})} = \mathbb{C}[\beta_0^x, \gamma_0^{x'} | x \in V, x' \in V^*]^{sl(2,\mathbb{C})} \cong \emptyset(V \oplus V^*)^{sl(2,\mathbb{C})}$ as a $\partial-$ring. Using Lemma 3.5, we know that $P_0^{sl(2,\mathbb{C})}$ is finitely generated. Then $P^{\Theta_+} = P^{sl(2,\mathbb{C})\otimes\mathbb{C}[t]}$ is finitely generated as $\partial-$ring. To describe the generators of $S(V)^{\Theta_+}$, we need to describe the generators of P^{Θ_+} by the Hilbert series of $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})^*$.

4 The Hilbert series and The generators of $P_0^{sl(2,\mathbb{C})}$

In this section, we shall calculate the Hilbert series of $P_0^{sl(2,\mathbb{C})}$, then give the generators of $P_0^{sl(2,\mathbb{C})}$, which are also the generators of P^{Θ_+} as $\partial-$ ring. Here, we refer to the related definitions and results in [13].

Definition 4.1. Let G be a subgroup of general linear group $GL(n)$, $T = \mathbb{C}[x_1, x_2, \cdots, x_n]$ be the polynomial ring, G has an action on T, denoted by T^G for the ring of invariants of G. $T = \bigoplus_{d\geq 0} T_d$, where $T_d \subset T$ is the subspace of homogeneous polynomials of degree d, then $T^G = \bigoplus_{d\geq 0} T^G \cap T_d$. There is a formal power series in an indeterminate t

$$P(t) = \sum_{d\geq 0} dim(T^G \cap T_d)t^d \in \mathbb{Z}[[t]] \tag{4.1}$$

is called the Hilbert series of the grading ring T^G.

In terms of the Hilbert series, there are the two important theorems.

Lemma 4.2. *([13]) If T^G is generated by homogeneous polynomials $f_1, f_2, \cdots, f_r$ of degree $d_1, d_2, \cdots, d_r$, then the Hilbert series of T^G is the power series expansion at $t = 0$ of rational function*

$$P(t) = \frac{F(t)}{(1-t^{d_1})(1-t^{d_2})\cdots(1-t^{d_r})} \tag{4.2}$$

for some $F(t) \in \mathbb{Z}[t]$.

Let V be any $n-$ dimensional representation of $sl(2,\mathbb{C})$, consider the induced action of $sl(2,\mathbb{C})$ on the polynomial ring $T(V) = \mathbb{C}[x_1, x_2, \cdots, x_n]$

of functions on V. Let $a_1, a_2, \cdot, a_n \in \mathbb{Z}$ be the weights (not necessarily distinct) of $sl(2,\mathbb{C})$ in the weight-space decomposition of V, the function

$$P(q;t) = \frac{1}{(1-q^{a_1}t)(1-q^{a_2}t)\cdots(1-q^{a_n}t)} = det\left(I_V - t\begin{pmatrix} q & 0 \\ 0 & q^{-1}\end{pmatrix}_V\right)^{-1} \tag{4.3}$$

is called the $q-$Hilbert series of the representation V. There is the relations between $q-$Hilbert series and Hilbert series as follows

Proposition 4.3. *([13]) The invariant ring $T(V)^{sl(2,\mathbb{C})}$ has Hilbert series*

$$P(t) = Res_{q=0}(q-q^{-1})P(q;t).$$

Equivalently, if $P(q;t) = \sum\limits_{m\in\mathbb{Z}} c_{(m)}(t)q^m$, then $P(t) = c_{(0)}(t) - c_{(-2)}(t)$.

Next, we shall compute the Hilbert series of $P_0^{sl(2,\mathbb{C})} = \mathbb{C}[\beta_0^e, \beta_0^f, \beta_0^h, \gamma_0^{e'}, \gamma_0^{f'}, \gamma_0^{h'}]^{sl(2,\mathbb{C})}$. Take $V = sl(2,\mathbb{C})$ is the adjoint representation of $sl(2,\mathbb{C})$, then $P_0^{sl(2,\mathbb{C})}$ is the the polynomial ring of functions on $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})^*$. Obviously, we know that $\{2,2,0,0,-2,-2\}$ is the set of all weights in the weight-space decomposition of the representation $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})^*$. Associated to the definition of $q-$ Hilbert series, we obtain the $q-$ Hilbert series of the representation $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})^*$ as follows

$$P(q;t) = \frac{1}{(1-q^2t)^2(1-t)^2(1-q^{-2}t)^2} \tag{4.4}$$

By Proposition 4.3 and some calculus, we can give the Hilbert series of the representation $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})^*$.

Proposition 4.4. *For the representation $sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})^*$, the Hilbert series of the invariant ring $P_0^{sl(2,\mathbb{C})} = \mathbb{C}[\beta_0^e, \beta_0^f, \beta_0^h, \gamma_0^{e'}, \gamma_0^{f'}, \gamma_0^{h'}]^{sl(2,\mathbb{C})}$ is*

$$P(t) = \frac{1}{(1-t^2)^3} \tag{4.5}$$

Proof. According to the Proposition 4.3, we need to write $P(q;t)$ as a formal series $\sum\limits_{m\in\mathbb{Z}} c_{(m)}(t)q^m$. Since the $q-$ Hilbert series

$$P(q;t) = \tfrac{1}{(1-q^2t)^2(1-t)^2(1-q^{-2}t)^2} = \tfrac{1}{(1-t)^2}\left(\tfrac{q^4}{(1-q^2t)^2(q^2-t)^2}\right)$$

After calculus, $P(q;t)$ can be decomposed into the following form

$$P(q;t) = \tfrac{1}{(1-t)^2}\left(\tfrac{2t^2}{(1-t^2)^3}\tfrac{1}{1-q^2t} + \tfrac{1}{(1-t^2)^2}\tfrac{1}{(1-q^2t)^2} + \tfrac{2t}{(1-t^2)^3}\tfrac{1}{q^2-t} + \tfrac{t^2}{(1-t^2)^2}\tfrac{1}{(q^2-t)^2}\right)$$

Using the following expansions of rational functions

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n; \frac{1}{1+u} = \sum_{n=0}^{\infty}(-1)^n u^n; \frac{1}{(1-u)^2} = \sum_{n=0}^{\infty}(n+1)u^n,$$

we get the following expansion

$$
\begin{aligned}
P(q;t) = \ & \tfrac{1}{(1-t)^2}\Big(\tfrac{2t^2}{(1-t^2)^3}\sum_{n=0}^{\infty} t^n q^{2n} + \tfrac{1}{(1-t^2)^2}\sum_{n=0}^{\infty}(n+1)t^n q^{2n} \\
& + \tfrac{2t}{(1-t^2)^3}\sum_{n=0}^{\infty} t^n q^{-2n-2} + \tfrac{t^2}{(1-t^2)^2}\sum_{n=0}^{\infty}(n+1)t^n q^{-2n-4}\Big) \\
& = \sum_{m\in\mathbb{Z}} c_{(m)}(t) q^m
\end{aligned}
$$

Hence there are

$$
C_{(0)}(t) = \tfrac{1}{(1-t)^2}\left(\tfrac{2t^2}{(1-t^2)^3} + \tfrac{1}{(1-t^2)^2}\right) = \tfrac{1}{(1-t)^2}\tfrac{1+t^2}{(1-t^2)^3}; C_{(-2)}(t) = \tfrac{1}{(1-t)^2}\tfrac{2t}{(1-t^2)^3}.
$$

Finally, we obtain the Hilbert series

$$
\begin{aligned}
P(t) \ & = C_{(0)}(t) - C_{(-2)}(t) = \tfrac{1}{(1-t)^2}\left(\tfrac{1+t^2}{(1-t^2)^3} - \tfrac{2t}{(1-t^2)^3}\right) \\
& = \tfrac{1}{(1-t^2)^3}.
\end{aligned}
$$

By Lemma 4.2, and Proposition 4.4, $P_0^{sl(2,\mathbb{C})} = \mathbb{C}[\beta_0^e, \beta_0^f, \beta_0^h, \gamma_0^{e'}, \gamma_0^{f'}, \gamma_0^{h'}]^{sl(2,\mathbb{C})}$ has three generators as a ring and these generators are all homogeneous elements of degree 2. and using Lemma 3.4, these generators are also generators of P^{Θ_+} as $\partial-$ring.

Proposition 4.5. *There are the following three elements of degree 2*

$$
v^e = 4(\gamma_0^{h'}\gamma_0^{h'} + \gamma_0^{e'}\gamma_0^{f'}), v^f = -\frac{1}{16}(\beta_0^h\beta_0^h + 4\beta_0^e\beta_0^f), v^h = \beta_0^h\gamma_0^{h'} + \beta_0^e\gamma_0^{e'} + \beta_0^f\gamma_0^{f'}, \tag{4.6}
$$

belong to $P_0^{sl(2,\mathbb{C})}$.

Proof. According to the Lemma 2.8, we know that $v^e(z), v^f(z), v^h(z) \in S(V)^{\Theta_+}$ with the same degree 2. If $V = \mathbf{g} = sl(2,\mathbb{C})$, then we can get an orthonormal basis $\{\frac{h}{2\sqrt{2}}, \frac{e+f}{2\sqrt{2}}, \frac{e-f}{2\sqrt{-2}}\}$ with respect to the Killing form K of $sl(2,\mathbb{C})$, the corresponding dual basis is $\{2\sqrt{2}h', \sqrt{2}(e'+f'), \sqrt{-2}(e'-f')\}$. Hence there are

$$
\begin{aligned}
v^e(z) \ & = \tfrac{1}{2}(:\gamma^{2\sqrt{2}h'}(z)\gamma^{2\sqrt{2}h'}(z): + :\gamma^{\sqrt{2}(e'+f')}(z)\gamma^{\sqrt{2}(e'+f')}(z): \\
& \quad + :\gamma^{\sqrt{-2}(e'-f')}\gamma^{\sqrt{-2}(e'-f')}) \\
& = 4(:\gamma^{h'}(z)\gamma^{h'}(z): + :\gamma^{e'}(z)\gamma^{f'}(z):),
\end{aligned}
$$

$$
\begin{aligned}
v^f(z) \ & = -\tfrac{1}{2}(:\beta^{\frac{h}{2\sqrt{2}}}(z)\beta^{\frac{h}{2\sqrt{2}}}(z): + :\beta^{\frac{e+f}{2\sqrt{2}}}(z)\beta^{\frac{e+f}{2\sqrt{2}}}(z): \\
& \quad + :\beta^{\frac{e-f}{2\sqrt{-2}}}(z)\beta^{\frac{e-f}{2\sqrt{-2}}}(z):) \\
& = -\tfrac{1}{16}(:\beta^h(z)\beta^h(z): + 4:\beta^e(z)\beta^f(z):)
\end{aligned}
$$

$$
\begin{aligned}
v^h(z) \ & =: \beta^{\frac{h}{2\sqrt{2}}}(z)\gamma^{2\sqrt{2}h'}(z) + :\beta^{\frac{e+f}{2\sqrt{2}}}(z)\gamma^{\sqrt{2}(e'+f')}(z): \\
& \quad + :\beta^{\frac{e-f}{2\sqrt{-2}}}(z)\gamma^{\sqrt{-2}(e'-f')}(z): \\
& =: \beta^h(z)\gamma^{h'}(z): + :\beta^e(z)\gamma^{e'}(z): + :\beta^f(z)\gamma^{f'}(z)
\end{aligned}
$$

Denote by $\mathcal{A} = \widehat{\psi}(\O(sl(2,\mathbb{C}), -\frac{3}{8}K))$. Since $gr(\mathcal{A}) \hookrightarrow gr(S(V)^{\Theta_+}) \hookrightarrow gr(S(V))^{\Theta_+} = P^{\Theta_+}$, so $\phi_2(v^u(z)) \in P_0^{sl(2,\mathbb{C})}$, where $u = e, f, h$, hence we know v^e, v^f, v^h belong to $P_0^{sl(2,\mathbb{C})}$.

As we expect, there is the following conclusion

Proposition 4.6. *three elements* $v^e = 4(\gamma_0^{h'}\gamma_0^{h'} + \gamma_0^{e'}\gamma_0^{f'}), v^f = -\frac{1}{16}(\beta_0^h\beta_0^h + 4\beta_0^e\beta_0^f), v^h = \beta_0^h\gamma_0^{h'} + \beta_0^e\gamma_0^{e'} + \beta_0^f\gamma_0^{f'}$, *are algebraically independent in* $P_0^{sl(2,\mathbb{C})}$.

Proof. Assume that there is a non-zero polynomial $H(y_1, y_2, y_3) \in \mathbb{C}[y_1, y_2, y_3]$ such that $H(v^e, v^f, v^h) = 0$. We can restrict to the subspace: $\gamma_0^{e'} = \beta_0^e = 0; \gamma_0^{f'} = \beta_f^0$, then there are $v^e = 4\gamma_0^{h'}\gamma_0^{h'}; v^f = -\frac{1}{16}\beta_0^h\beta_0^h; v^h = \beta_0^h\gamma_0^{h'} + \beta_0^f\gamma_0^{f'}$, then define a map

$$\mu : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$$
$$(\beta_0^f, \beta_0^h, \gamma_0^{h'}) \longmapsto (v^e, v^f, v^h) = (4\gamma_0^{h'}\gamma_0^{h'}, -\frac{1}{16}\beta_0^h\beta_0^h, \beta_0^h\gamma_0^{h'} + \beta_0^f\gamma_0^{f'}).$$

Since one can separate variables for the map μ, so it is a surjective, hence v^e, v^f, v^h don't satisfy any identity relation, of course, there isn't any polynomial relation of v^e, v^f, v^h, then there is a contradiction. Therefore, it dosen't exist non-zero polynomial $H(y_1, y_2, y_3)$ such that $H(v^e, v^f, v^h) = 0$. Three elements $v^e = 4(\gamma_0^{h'}\gamma_0^{h'} + \gamma_0^{e'}\gamma_0^{f'}), v^f = -\frac{1}{16}(\beta_0^h\beta_0^h + 4\beta_0^e\beta_0^f), v^h = \beta_0^h\gamma_0^{h'} + \beta_0^e\gamma_0^{e'} + \beta_0^f\gamma_0^{f'}$ are algebraically independent in $P_0^{sl(2,\mathbb{C})}$.

5 The Main Results about the Commutant $S(V)^{\Theta_+}$

According to given propositions in section 4, if $\mathbf{g} = sl(2,\mathbb{C}) = V$, we obtain some results as follows.

For the commutant $S(V)^{\Theta_+}$, we know that the weight-zero subspace $S(V)_0^{\Theta_+} = Sym(V^*)^{sl(2,\mathbb{C})}$ by Lemma 2.6.

Proposition 5.1. *The conformal weight-zero subspace* $S(V)_0^{\Theta_+}$ *is generated only by the element* $:\gamma^{e'}(z)\gamma^{f'}(z): + :\gamma^{h'}(z)\gamma^{h'}(z):$.

Proof. At first, we know $S(V)_0^{\Theta_+} = Sym(V^*)^{sl(2,\mathbb{C})} \cong \mathbb{C}[\gamma^{e'}, \gamma^{f'}, \gamma^{h'}]^{sl(2,\mathbb{C})}$. Since the Hilbert series of $V = sl(2,\mathbb{C})$ is $P(t) = \frac{1}{1-t^2}$ in [13], so the invariant ring $\mathbb{C}[\gamma^{e'}, \gamma^{f'}, \gamma^{h'}]^{sl(2,\mathbb{C})}$ is only generated by an element of degree 2. It is easy to check $\gamma^{e'}\gamma^{f'} + \gamma^{h'}\gamma^{h'}$ is an element with degree 2 in $\mathbb{C}[\gamma^{e'}, \gamma^{f'}, \gamma^{h'}]^{sl(2,\mathbb{C})}$. Thus $S(V)_0^{\Theta_+}$ is generated by the element $:\gamma^{e'}(z)\gamma^{f'}(z): + :\gamma^{h'}(z)\gamma^{h'}(z):$ as an algebra.

Theorem 5.2. *The invariant ring* $P_0^{sl(2,\mathbb{C})}$ *is generated by three elements* $v^e = 4(\gamma_0^{h'}\gamma_0^{h'} + \gamma_0^{e'}\gamma_0^{f'}), v^f = -\frac{1}{16}(\beta_0^h\beta_0^h + 4\beta_0^e\beta_0^f), v^h = \beta_0^h\gamma_0^{h'} + \beta_0^e\gamma_0^{e'} + \beta_0^f\gamma_0^{f'}$, *therefore the invariant ring* P^{Θ_+} *is generated by three elements* v^e, v^f, v^h *as* $\partial-$*ring.*

Proof. Using Lemma 3.4 and Proposition 4.4, 4.5, 4.6, we can get above result immediately.

Let $\mathcal{A}$ be the image of the homomorphism $\widehat{\psi}$ in Lemma 2.3. i.e. $\mathcal{A} = \widehat{\psi}(\emptyset(sl(2,\mathbb{C}), -\frac{3}{8}K))$, we have

Theorem 5.3. *the commutant $S(V)^{\Theta_+}$ is generated strongly by three elements*

$$v^e(z) = 4(:\gamma^{h'}(z)\gamma^{h'}(z): + :\gamma^{e'}(z)\gamma^{f'}(z):),$$

$$v^f(z) = -\frac{1}{16}(:\beta^h(z)\beta^h(z): +4 :\beta^e(z)\beta^f(z):),$$

$$v^h(z) =: \beta^h(z)\gamma^{h'}(z): + :\beta^e(z)\gamma^{e'}(z): + :\beta^f(z)\gamma^{f'}(z),$$

and there is $S(V)^{\Theta_+} = \mathcal{A}$.

Proof. Since $\phi_2(v^e(z)) = v^e; \phi_2(v^f(z)) = v^f; \phi_2(v^h(z)) = v^h$ and we have known $v^e(z), v^f(z), v^h(z) \in S(V)^{\Theta_+}$, so $v^e, v^f, v^h \in gr(\mathcal{A}) \hookrightarrow gr(S(V)^{\Theta_+})$. By Theorem 5.2, P^{Θ_+} is generated by three elements v^e, v^f, v^h as $\partial-$ring, hence $gr(S(V)^{\Theta_+}) = gr(S(V))^{\Theta_+} = P^{\Theta_+}$. According to Lemma 3.1, we know $S(V)^{\Theta_+}$ is generated strongly by three elements $v^e(z) = 4(:\gamma^{h'}(z)\gamma^{h'}(z): + :\gamma^{e'}(z)\gamma^{f'}(z):)$, $v^f(z) = -\frac{1}{16}(:\beta^h(z)\beta^h(z): +4 :\beta^e(z)\beta^f(z):)$, $v^h(z) =: \beta^h(z)\gamma^{h'}(z): + :\beta^e(z)\gamma^{e'}(z): + :\beta^f(z)\gamma^{f'}(z)$.

However, A is also generated by these three elements $v^e(z), v^f(z), v^h(z)$ as a subalgebra of $S(V)^{\Theta_+}$, so $S(V)^{\Theta_+} = \mathcal{A}$.

Theorem 5.4. *The commutant $S(V)^{\mathcal{A}_+} = Com(\mathcal{A}, S(V))$ and $\mathcal{A}$ form a Howe pair in vertex algebra context.*

Proof. Since $S(V)^{\Theta_+} = \mathcal{A}$, then $\Theta = \widehat{\rho}(\emptyset(sl(2,\mathbb{C}), -K)$ is a subalgebra of $S(V)^{\mathcal{A}_+}$, thus there is

$$\mathcal{A} \subset Com(S(V)^{\mathcal{A}_+}, S(V)) \subset Com(\Theta, S(V)) = S(V)^{\Theta_+} = \mathcal{A},$$

so we have $\mathcal{A} = Com(S(V)^{\mathcal{A}_+}, S(V))$, i.e, $(\mathcal{A}, S(V)^{\mathcal{A}_+})$ forms a Howe pair in vertex algebra context.

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